KING ABDULAZIZ UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ENTRANCE TEST FOR PH.D. MATHEMATICS
19th Jumada Awal, 1434 / 31st March 2013
Maximum Time: Three Hours

*Name:*____

I.D. #: ____

INSTRUCTIONS:
1. Solve all Seven questions given in the exam.
2. All questions are equally weighted.
3. Write complete steps in a proof.
4. To disprove a statement, provide a counter example.

	Grading	
	Assigned	Obtained
Q.1	(10)	
Q.2	(10)	
Q.3	(10)	
Q.4	(10)	
Q.5	(10)	
Q.6	(10)	
Q.7	(10)	
Total:	(70)	

Q.1.

(a) Prove that any convergent sequence x_n of real numbers with

$$\lim x_n = x$$

is Cauchy.

Q.1.

(b) Let X be a metric space with distance d.
Define a continuous function $f : X \to \mathbb{R}$.

Q.2.

(a) Let X be a Banach space and $\mathbb{R}$ the usual normed space. Show that

$$||(x,\alpha)|| \ = ||x|| \ + \ |\alpha|$$

is a complete norm on $X \times \mathbb{R}$.

Q.2.

(b) Let $X = \mathbb{R}^3$. Show that the mapping $T : X \to \mathbb{R}$ defined by

$$T(x) = x_1 + x_2 + x_3,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, is a continuous linear operator for which

$$||T|| = \sqrt{3}.$$

Q.3.

Prove or disprove the following statements.

(**a**) Any two groups of order p, where p is a prime number, are isomorphic.

(**b**) $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ are isomorphic abelian groups.

Q.3.

(**c**) The group of quaternions $Q_8 = \{\pm 1,\ \pm i,\ \pm j,\ \pm k\}$ is a cyclic group.

Q.4.

Let R be a commutative ring with identity.

An ideal I of R is called a prime ideal if for any pair of elements $a, b \in I$,

$$ab \in I \Longrightarrow a \in I \quad \text{or } b \in I.$$

(**a**) Prove that: Every maximal ideal of R is a prime ideal.

Q.4.

(**b**) Let P be an ideal of R. Then prove that:

P is prime if and only if R/P is an integral domain.

Q.5.

Define $\mathcal{T} \subset \mathcal{P}(\mathbb{R})$ as follows:

A subset U of $\mathbb{R}$ is in $\mathcal{T}$ if and only if either $U = \Phi$; or $\mathbb{R} \backslash U$ is countable.

(a). Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$. This topology is called the *countable complement topology.*

(b). Let $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{I}$ denote the irrationals. Calculate in $(\mathbb{R}, \mathcal{T})$ each of the following:

(1) $int(\mathbb{I})$.

(2) $\overline{\mathbb{Q}}$.

Q.5.

(c). Prove or disprove: $(\mathbb{R}, \mathcal{T})$ is T_2.

(**d**). Prove or disprove: $(\mathbb{R}, \mathcal{T})$ is second countable.

Q. 6.

Let us consider the Cauchy Problem (CP)

$$x' = f(t,x), \qquad x(t_0) = x_0,$$

where

$$f : \overline{D} \to \mathbb{R}, D = \{(t,x) | |t - t_0| \leq a,\ |x - x_0| \leq b\}.$$

Peano says that a solution always exists in

$$|t - t_0| \leq \alpha,\ \text{where } \alpha = min\{a, \frac{b}{M}\},\ \text{and } M = max_{(t,x)\in\overline{D}}|f(t,x)|,$$

if $f(t,x)$ is continuous.

(a) Give an example to show that this condition is sufficient but not necessary.

(b) Show, by giving an example, that (CP) may still have a solution and this solution may be unique, however, f does not satisfy Lipschtiz condition. Justify your answer.

Q. 7.

Solve the following system of nonlinear equations using Newton's method & Jacobian matrix.

$$3x_1 - \cos(x_1 x_2) - \frac{1}{2} = 0,$$

$$4x_1^2 - 625x_2^2 + 2x_2 - 1 = 0,$$

$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$